Relationship between sampling rate and quantization noise

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Abstract.- Few works have been done about the dependency of the quantization noise with the sampling rate for uniform quantizers. Some of these works have considered the problem from a deterministic point of view while others study it from a stochastic one, having explained the noise behavior in some specific cases. By using computer simulations with a sinusoidal input signal, here we show that the quantization noise spectrum can show a discrete or complex structure depending on the sampling rate used. The results confirm that an **integer** ratio between the sampling rate (f_r) and the frequency of the input signal (f_s) produces a quantization noise with components in odd harmonics of the signal frequency. If there is not such **integer** ratio between f_r and f_s, then the quantization noise can present a stochastic structure for some rational ratios. An additional result is that the phase of the input signal can also modify the magnitude of the spectral components of the quantization noise. These results show that the quantization noise is clearly dependent on the input signal and the sampling rate.

I. INTRODUCTION

Since the development of pulse count modulation systems, there has been a large interest in the study of the structure and behavior of the quantization noise generated by an ideal uniform staircase quantizer. However, this work has shown to be difficult due to the nonlinearity of the system. By this reason, it is generally accepted the simplified model in which the quantizer noise is a uniformly distributed white noise, independent on the signal driven the quantizer. This approximation can lead to inaccurate predictions in the results as has been signaled in some references [1] [2]. By this reason some researchers have studied the quantization noise behavior either from a deterministic or from a stochastic point of view. Based on their models, partial descriptions of this behavior have been obtained.

In this paper, after presenting a brief revision of several classic papers that analyze the quantization noise, the sampling and quantization processes are analyzed in some particular cases, where the generally accepted assumption of additive white noise is clearly not valid and where other previous results need to be reconsidered. By using computer simulations, the noise spectra will be obtained in these case and some conclusions will be presented.

II. BACKGROUND

The first two works that studied quantization noise in an ideal uniform quantizer, were those written by W. R. Bennet [3] and Clavier, Panter and Grieg [4]. Bennet established that if quantum values have sufficiently close spacing, the quantized wave can be indistinguishable from the original one. In that work, he also commented that the analog-to-digital conversion of a single- or double-frequency signal generated a ragged spectrum or a spectrum with discrete frequency components, although he did not analyze the distribution of those discrete components. He studied a signal having energy uniformly distributed throughout a definite frequency band or a signal with a large number of input components. In these cases, he obtained a quantization noise with a uniform spectrum or at least a spectrum with very smoothed irregularities.

In the second work, Clavier *et al.* made a complete analysis of the quantizer noise from a deterministic point of view. They calculated the distortion produced by a step function when the input signal was a sine wave, considering an integer number of samples in each period of the signal. Under these conditions they obtained that the distortion components were always odd harmonics of the frequency of the input signal.

In 1960, Widrow [5] presented what is referred to as the quantization theorem, and the uniform white noise assumption gained a wide popularity being now commonly used. Seventeen years after Widrow's work, Sripad and Snyder [6], using a stochastic method, established a necessary and sufficient condition to model the output of a quantizer as an infinite-precision input plus an additive, uniform, white noise. Their result expanded the class of input distributions for which the quantization noise is white, as Widrow had established that the quantization noise density is uniform if the input has a band-limited characteristic function. Sripad and Synder included other input functions like variables with triangular or Gaussian density functions.

Next, in 1981, Claasen and Jongepier [7] considered a model for the error spectrum which only required knowledge of the amplitude distribution of the derivative of the input signal. With this model, they determined that when the sinusoidal signal has sufficient variation, it can be assumed that the quantization error has a white spectrum. Their model

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also predicted, for signals with not sufficient variations, poles in the spectrum of the noise at discrete frequencies $w = 2pXw_0$ being the input signal $x(t) = X \sin(w_0 t)$. In 1999, a deterministic approach to the problem of quantization was proposed by Bellan *et al*. [8]. They confirmed the results presented by Claasen and Jongepier, but they did not mention the work by Clavier *et al*. nor that by Gray about the presence of odd harmonics of the signal fundamental frequency. In the study by Bellan, the effect of the phase appears in a complex exponential factor, thus not affecting the magnitude spectrum of the quantization error.

Finally, Gray in his work published in 1990 [1], confirmed the result obtained by Clavier *et al*. about the structure of the quantization noise spectrum, but he concluded that the odd harmonics of the frequency of the sinusoidal signal appear in the noise spectrum independently on the value of that frequency and its relation with the sampling frequency. He obtained an expression where the phase of the signal also appears in a complex exponential factor and therefore it does not affect the magnitude spectrum.

Then the aim of our work has been to confirm or to reconsider the conclusions reached by R.M. Gray in 1990 and by Bellan *et al*. in 1999. Additionally, we tried to call the attention about an important aspect in the process of analog to digital conversion: it is a dynamical process, with a strong dependency on the sampling rate.

III. METHOD

In order to study the behavior of the quantization noise produced in the process of uniform quantization of a sinusoidal input signal, computer simulations with MATLAB were performed, under different conditions. Computer simulations allowed us to use classical roundoff, avoiding completely the effect of other factors appearing in the operation of an analog-to-digital converter, like jitter, integral and differential non-linearity, thermal noise, etc.

In all the cases, the sampling process is simulated by generating a time variable with values determined by the specified sampling frequency and taking values of the sinusoidal signal at times determined by the time variable. Then the samples are quantized and the quantization error computed. Finally, the spectrum of the unquantized signal, the quantized signal and the error signal are computed. Here we present only two of several simulations realized.

1. Integer and non-integer relation between sampling and signal frequencies.

The aim of this simulation is to show the behavior of quantization noise when the relation between the sampling and the signal frequencies is an integer or a non-integer number. The simulation has been done following the steps:

- A sinusoidal signal with zero phase is generated with the maximum resolution permitted by the computer and oversampled at 40 Ksamples/s. On the whole, 4096 samples are produced.
- The signal is then quantified to 188 levels.

This procedure is repeated for signals with frequencies 800 Hz, 800.1 Hz and 801 Hz. That means an oversampling factor of 50, 49.99 and 49.94, respectively.

2. Relation between the signal phase and the structure of the spectrum of the quantized signal.

The procedure followed in the previous simulation is repeated here for the signal with a frequency of 800 Hz, once with a zero phase and another with a phase of $\frac{p}{2}$.

IV. RESULTS

1. Integer and non-integer relation between sampling and signal frequencies.

As shown in Fig. 1 the spectrum of the unquantized signal (quantization given only by MATLAB' resolution) presents only one peak that corresponds to the signal frequency.

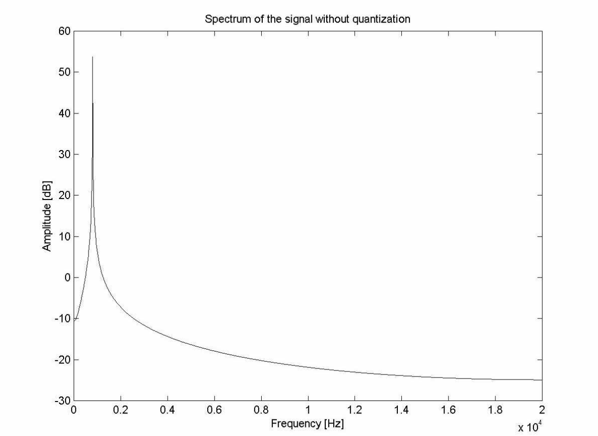

Fig. 1. Spectrum of the unquantized sinusoidal signal (frequency 800 Hz, phase equal to zero, sampling rate 40 Ksamples/s).

Fig. 2. presents the spectrum of the same sinusoidal signal but now quantized with 188 levels. Here the spectrum of the sinusoid has several peaks in addition to the fundamental one. These peaks appear at all odd harmonics of the input signal fundamental frequency. This result agrees with that obtained by Gray [1] but here an integer relation (50) between the sampling rate and the frequency of the sinusoid is used. If the frequency of the signal is slightly changed (49.99 instead of 50), then the power of the odd harmonics starts spreading in the frequency band establis hed by the sampling rate as shown in Fig. 3. This is an aspect that has not been presented before (at least at our knowledge). If the ratio between frequencies further differs from the integer value (now 49.93), the noise power spreads along the entire Nyquist band as is shown in Fig. 4. It must be remarked that for a large number of rational ratios, the spectrum acquires the structure that Gray presented, but this is not the case for all the ratios.

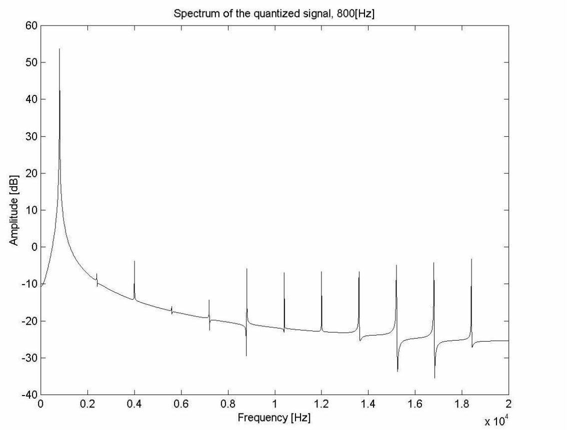

Fig.2. Spectrum of the sinusoidal signal (800 Hz, zero phase, sampling rate 40 Ksamples/s) quantized using 188 levels.

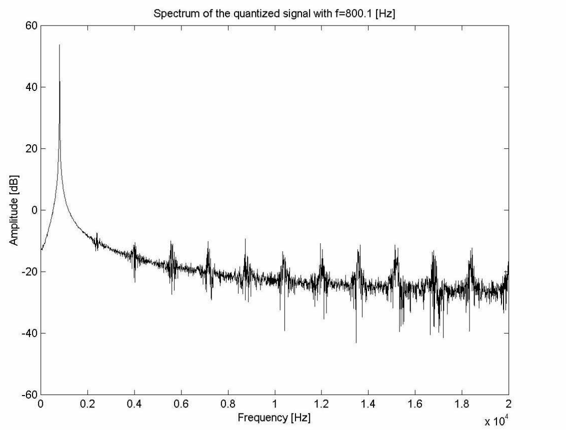

Fig. 3. Spectrum of a quantized sinusoidal signal of frequency 800.1 Hz, sampled at 40 Ksamples/s.

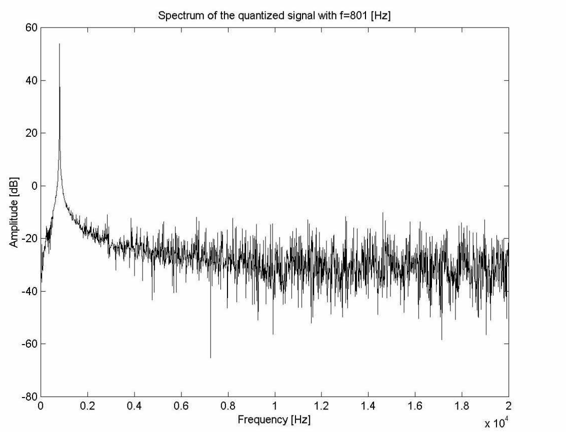

Fig. 4. Spectrum of a quantized sinusoidal signal of frequency 801 Hz, sampled at 40 Ksamples/s.

2. Relation between the signal phase and the spectrum of the quantized sinusoidal signal.

If the phase of the sinusoidal signal to be quantized is modified, the magnitude of the odd harmonics changes and, in some cases, some of the harmonics can disappear. This effect can be appreciated comparing figures 2 and 5. The second and seventh odd harmonic have disappeared in fig. 5 or at least absorbed by the background level.

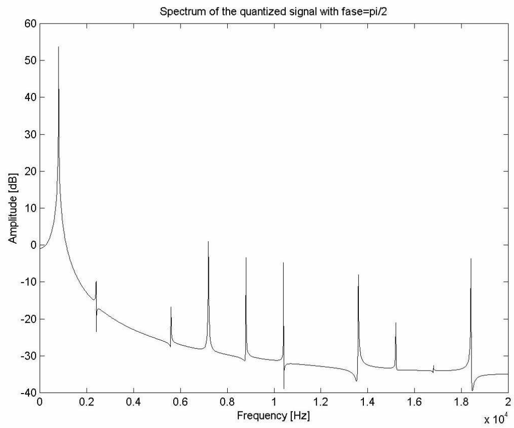

Fig. 5. Spectrum of the quantized sinusoidal signal of frequency 800 Hz and phase $\frac{p}{2}$.

Finally, it is important to say that similar results can be observed using undersampling. In the cases presented here, the possible aliasing could not be appreciated because high (aliased) harmonics would appear at frequencies of low harmonics. This aliasing would appear for example if using a sinusoid of frequency 37.5KHz and a sampling rate of 4 Msamples/s.

V. CONCLUSIONS

This work shows, as Gray and Claasen-Jongepier also showed, that it is not correct to model the output of a quantifier by an infinite precision signal plus additive white noise. When a sinusoidal signal is sampled and quantized, if the signal and the sampling frequencies have an integer, or in a large number of cases, a rational relation in the continuous time, the quantization error has odd harmonics of the input signal. This fact has also been observed by Kester [2] as he remarks that during the evaluation of analog-to-digital converters, discrete frequencies can appear in harmonics of the frequency of the input signal. But Kester does not mention that the quantization noise concentrates only in the odd harmonics, and that this accumulation can also appear for some other rational ratio different from the integer one.

The harmonics disappear gradually or their power spreads in the whole band defined by the sampling frequency, as the signal or sampling frequencies change. This result is in contradiction with Gray's results, as he says that the presence of harmonics is independent on the signal frequency or the relation between signal and sampling frequency. Gray's analysis and Claasen-Jongepier's are still valid if their initial conditions are correct: if the quantization error is periodic or almost periodic, but this is only true, as we have shown when the relation between the input and the sampling frequencies is an integer or, in some cases, a rational number. The quantization noise presents a chaotic behavior because for some values of the relation between the signal and the sampling frequencies presents a very well defined structure, while for other values this structure is lost.

The dependency with the signal phase should also be remarked. The harmonics amplitudes depend on certain degree of the signal phase, although the total error energy is obviously constant. Both Bellan *et al*. and Gray introduced the phase factor as a complex exponential, so it did not affect the magnitude spectrum.

The presence of these harmonic components in the quantization error has to be considered when designing digital system that process analog signals. Although they are usually undesirable as they distort the digital signal, the effect could also be used to evaluate ADCs: the quantization noise could be well localized and any additional noise would be due to other sources.

VI. ACKNOWLEGMENTS

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